

Final exam Linear Algebra II

Thursday 03/04/2025, 15:00–17:00

1 (9 = 3 + 3 + 3 pts)

orthogonal complement

Let P_2 denote the \mathbb{R} -vector space of all real polynomials of degree at most 2, that is,

$$P_2 = \{p \mid p(x) = p_0 + p_1x + p_2x^2 \text{ where } p_0, p_1, p_2 \in \mathbb{R}\}.$$

For any $p \in P_2$, define the vector $\tilde{p} \in \mathbb{R}^3$ by

$$\tilde{p} := \begin{bmatrix} p(0) \\ p(1) \\ p(-1) \end{bmatrix}.$$

It can be shown that $\langle p, q \rangle := \tilde{p}^\top \tilde{q}$ defines an inner product on P_2 .

- (a) Compute a nonzero polynomial in P_2 that is orthogonal to both 1 and x .

A polynomial p is called *even* if $p(x) = p(-x)$ for all x , and *odd* if $p(-x) = -p(x)$ for all x . Let $\mathcal{E} \subset P_2$ be the subspace of all even polynomials and $\mathcal{O} \subset P_2$ the subspace of all odd polynomials.

- (b) Show that \mathcal{E} and \mathcal{O} are orthogonal.
(c) Prove that $\mathcal{O} = \mathcal{E}^\perp$.

2 (9 = 3 + 3 + 3 pts)

dimension theorem, adjoint

Let V be a finite dimensional \mathbb{R} -inner product space, with inner product $\langle \cdot, \cdot \rangle$. Suppose $T: V \rightarrow V$ is an \mathbb{R} -linear map, and denote the adjoint of T by T^* .

- (a) Prove that $\text{Ker}(T^*) = T(V)^\perp$.
(b) Show that the adjoint of T^* is the map T .
(c) Prove that T is bijective $\iff T^*$ is bijective.

3 (9 = 3 + 3 + 3 pts)

self-adjoint, Gram-Schmidt, projection

In this problem, P denotes the inner product space consisting of all real polynomials in the variable t . The inner product on P is defined by $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$.

- (a) For any $p(t) \in P$, let $T: P \rightarrow P$ be the \mathbb{R} -linear map given by $T(f(t)) = p(t)f(t)$ (so, T is “multiplication by $p(t)$ ”). Show that T is self-adjoint.

- (b) Given is the subspace $Q \subset P$ defined as

$$Q = \{f(t) \in P : \deg(f(t)) \leq 2 \text{ and } f(1) = 0\}.$$

Use the fact that $t - 1 \in Q$ and $4t^2 - 5t + 1 \in Q$ are perpendicular, to construct an orthonormal basis for Q .

- (c) Now consider P and Q as in (b) as subspaces of $C([0, 1], \mathbb{R})$, the inner product space of all real continuous functions $[0, 1] \rightarrow \mathbb{R}$, with inner product $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$. Compute the orthogonal projection of the function $t \mapsto -1 + \sqrt{t}$ onto Q .

4 (9 = 6 + 3 pts)

singular value decomposition

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

- (a) Find a singular value decomposition of A .
- (b) Determine the distance $d(A, M_1)$ and compute a best rank-1 approximation of A .

4 pts free